Castillo, K.. (2014). On perturbed Szegö recurrences. *Journal of Mathematical Analysis and Applications*, 411, pp. 742-752.

DOI: 10.1016/j.jmaa.2013.10.014

On perturbed Szegő recurrences

Kenier Castillo [1]

Departamento de Matemática Aplicada, Universidade Estadual Paulista – UNESP, 15054-000, São José do Rio Preto, SP, Brazil

A B S T R A C T

The purpose of the present contribution is to investigate the effects of finite modifications of Verblunsky coefficients on Szegő recurrences. More precisely, we study the structural relations and the corresponding $\mathcal{C}$-functions of the orthogonal polynomials with respect to these modifications from the initial ones.

Keywords: Three-term recurrence relation, Orthogonal polynomials on the real line, Szeg ˝o recurrences, Orthogonal polynomials on the unit circle, Perturbed Verblunsky coefficients

1. Introduction

1.1. The real line case

A sequence of polynomials $\{p_n(x)\}_{n\geqslant 0}$ defined by the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad a_n > 0,\ b_n \in \mathbb{R}, \tag{1.1}$$

with initial conditions $p_{-1}(x) := 0$ and $p_0(x) := 1$, is said to be a sequence of orthonormal polynomials with respect to a measure supported on an infinite subset of the real line $I \subseteq \mathbb{R}$. Indeed, there exists a non-trivial probability measure $d\mu(x)$ such that

$$\int_I p_n(x)p_m(x)\,d\mu(x) = \delta_{n,m}, \quad m \geqslant 0,$$

where $\delta_{n,m}$ is the Kronecker delta. This result is known in the literature of orthogonal polynomials as Favard's theorem [3,26]. In general, the orthogonality measure $d\mu(x)$ is not unique, but a sufficient condition for the uniqueness is that the recurrence coefficients $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ are bounded [3].

It is very well known that the *orthogonal polynomials on the real line* (OPRL, in short) have useful properties in the solution of mathematical and physical problems. These solutions may be visualized as modifications of the original sequence of orthogonal polynomials. In [13], motivated by their potential application in boundary valued problems of finite-difference

E-mail addresses: kcastill@math.uc3m.es, kenier@ibilce.unesp.br.

[1] The author thanks the referee for careful reading and valuable comments. The author also wishes to express his thanks to F. Marcellán for suggesting the problem during the 1st Joint Conference of the Belgian, Royal Spanish and Luxembourg Mathematical Societies in Liège, Belgium. The research of the author was supported by CNPq Program/Young Talent Attraction, Ministério da Ciência, Tecnologia e Inovação of Brazil, Project 370291/2013-1 and Dirección General de Investigación, Ministerio de Economía y Competitividad of Spain, Grant MTM2012-36732-C03-01.

equations, the authors have studied the sequence of polynomials which satisfies a recurrence relation as (1.1) with new recurrence coefficients, $\{c_n\}_{n\geqslant 0}$ and $\{d_n\}_{n\geqslant 0}$, i.e.,

$$xq_n(x) = c_{n+1}q_{n+1}(x) + d_nq_n(x) + c_nq_{n-1}(x),$$

with initial conditions $q_{-1}(x) := 0$ and $q_0(x) := 1$, perturbed in a (generalized) co-dilated and/or co-recursive way, from now on, namely *co-polynomials on the real line* (COPRL, in short). In other words, the authors considered an arbitrary single modification of the recurrence coefficients as follows: either

$$c_j := \gamma a_j, \qquad c_m := a_m, \quad \gamma > 0,\ m \neq j, \qquad d_n := b_n, \quad \text{(co-dilated case)},$$

or

$$d_k := b_k + \xi, \qquad d_m := b_m, \quad \xi \in \mathbb{R},\ n \neq k, \qquad c_n := a_n, \quad \text{(co-recursive case)}.$$

Here, j and k are fixed non-negative integers numbers.

The study of the algebraic and analytic properties of the COPRL and its applications was initiated by Chihara [2] and later continued by several authors. Among others, the contributions of Marcellán [19,13], M a r o n i [4,18], and Ronveaux [17,18] are remarkable. For some applications, see [22,10,5,6]. More recently it has been generalized to recurrence relations of all orders [9]. It is worth pointing out that this kind of recurrence arises in the study of the Sobolev OPRL [7].

1.2. The unit circle case

The complex analogue of the theory of OPRL is naturally played by the *orthogonal polynomials on the unit circle* (OPUC, in short). Hence, a natural question arises here: What do co-polynomials mean on the unit circle? Once this is clarified, our focus will be to determine the properties of these polynomials from those of the original orthogonal polynomials. As far as we know, this problem has not been studied in the literature.

Surprisingly, the theory of orthogonal polynomials with respect to non-trivial probability measures supported on the unit circle $\partial\mathbb{D} = \{z \in \mathbb{C};\ |z| = 1\}$ has not been so popular until the mid-1980s. It was Szeg ˝o, in 1920 [24,25], motivated by his work on asymptotics of Toeplitz determinants [23], who first defined the sequence of orthonormal polynomials on the unit circle $\{\phi_n(z)\}_{n\geqslant 0}$ by

$$\int_{\partial\mathbb{D}} \phi_n(z)\overline{\phi_m(z)}\, d\sigma(z) = \delta_{n,m}, \quad m \geqslant 0, \tag{1.2}$$

where $d\sigma(z)$ is a non-trivial probability measure supported on the unit circle $\partial\mathbb{D}$, parametrized by $z = e^{i\theta}$, $\theta \in [-\pi, \pi)$. According to Fejér's theorem [20], t h e z e r o s o f $\phi_n(z)$ lie in $\mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C};\ |z| < 1\}$ is the open unit disk. Denoting by $\kappa_n > 0$ the leading coefficient of $\phi_n(z)$, the orthonormal polynomial of degree n, $\Phi_n(z) = \phi_n(z)/\kappa_n$ is the corresponding monic orthogonal polynomial.

We have seen that the OPRL satisfy a second order linear recurrence relation. Such a recurrence relation does not hold for the OPUC. Szeg ˝o s h o w e d [26] that $\{\Phi_n(z)\}_{n\geqslant 0}$ satisfies a first order recurrence relation (Szeg ˝o's recurrences),

$$\begin{bmatrix} \Phi_{n+1}(z) \\ \Phi^*_{n+1}(z) \end{bmatrix} = \mathbf{C}_n(z) \begin{bmatrix} \Phi_n(z) \\ \Phi^*_n(z) \end{bmatrix}, \quad \mathbf{C}_n(z) = \begin{bmatrix} z & -\overline{\alpha}_n \\ -\alpha_n z & 1 \end{bmatrix}, \tag{1.3}$$

with initial condition $\Phi_0(z) := 1$, where $\Phi^*_n(z) = z^n\overline{\Phi}_n(z^{-1})$ is the so-called reversed polynomial and $\mathbf{C}_n(z)$ is said to be a transfer matrix [20]. Notice that if we set $\Phi_n(z) := 0$ and $\Phi_n{}^*(z) := 0$ f o r $n \leqslant -1$, and $\alpha_{-1} := -1$ and $\alpha_n := 0$, for $n \leqslant -2$, then (1.3) holds for all $n \in \mathbb{Z}$.

In the engineering literature, the Szegő recurrences appear in the Levinson algorithm after its rediscovery in linear prediction theory [11]. The complex numbers $\{\alpha_n\}_{n\geqslant 0}$ where $\alpha_n = -\overline{\Phi_{n+1}(0)}$ (Geronimus' formula), are known as Verblunsky, Schur, Geronimus, or reflection coefficients, which satisfy

$$\alpha_n \in \mathbb{D}. \tag{1.4}$$

Influenced by the monograph of Simon [20], we refer to $\{\alpha_n\}_{n\geqslant 0}$ as Verblunsky coefficients.

In this context, there is an analogous of Favard's theorem, called in the contemporary literature as Verblunsky's Theorem [20,21].

Theorem 1.1 *(Verblunsky Theorem). (See [20,21].) Let $\{\zeta_n\}_{n\geqslant 0}$ be a sequence of complex numbers in $\mathbb{D}$. Then, there is a unique non-trivial probability measure $d\sigma(z)$ supported on the unit circle such that $\alpha_n = \zeta_n$.*

Based on the above theorem, the OPUC are completely determined by their Verblunsky coefficients. This fact reaffirms the need to study orthogonal polynomials associated with modifications of the original Verblunsky coefficients. In our opinion, one of the most interesting results in this direction appeared in [15]. In this work, Peherstorfer introduced and studied the so-called associated polynomials on the unit circle. Indeed, for a fixed positive integer number r, the associated polynomials of order r, $\{\Phi_n^{[r]}(z)\}_{n\geqslant 0}$, are generated by the shifted Verblunsky coefficients $\{\alpha_{n+r}\}_{n\geqslant 0}$ through the Szegő recurrences with initial condition $\Phi_0^{[r]}(z) := 1$.

If we replace in the Szegő recurrences the sequence $\{\alpha_n\}_{n\geqslant 0}$ by $\{-\alpha_n\}_{n\geqslant 0}$, then we obtain the sequence of second kind polynomials $\{\Omega_n(z)\}_{n\geqslant 0}$, that is a sequence of OPUC according to Verblunsky Theorem. These polynomials can be expressed in terms of the monic orthogonal polynomials with respect to the measure $d\sigma(z)$ as follows:

$$\Omega_n(z) = \int_{\partial\mathbb{D}} \frac{y+z}{y-z}\big(\Phi_n(y) - \Phi_n(z)\big)\, d\sigma(y). \tag{1.5}$$

Notice that $\deg \Omega_n(z) = n$ and $\Omega_n(z)$ is monic.

An important relation between the sequences $\{\Phi_n(z)\}_{n\geqslant 0}$ and $\{\Omega_n(z)\}_{n\geqslant 0}$ is given in [8],

$$\Phi_n(z)\Omega_n^*(z) + \Omega_n(z)\Phi_n^*(z) = 2z^n \prod_{j=0}^{n-1}\big(1 - |\alpha_j|^2\big) = 2z^n\kappa_n^{-2}, \quad n \geqslant 1. \tag{1.6}$$

We briefly outline the organization of the paper. In Section 2, we obtain an explicit expression of the *co-polynomials on the unit circle* (COPUC, in short) in terms of the original orthogonal polynomials. Similar results are also given simultaneously for the corresponding second kind polynomials. In Section 3, we analyze the pure rational spectral transformation for non-trivial $\mathcal{C}$-functions, associated with the COPUC, and we point out the relation with quadratic irrationalities. Further, we complete the classification of these transformations. Finally, in Section 4 we point out the connection with the real line case.

2. Structural relations

For a fixed non-negative integer number k, let us consider the perturbed Verblunsky coefficients $\{\beta_n\}_{n\geqslant 0}$ given by

$$\beta_n := \begin{matrix} \beta_k, & n = k; \\ \alpha_n, & \text{otherwise}. \end{matrix} \tag{2.7}$$

According to (1.4), in order to achieve a new valid sequence of Verblunsky coefficients, from now on we assume that $\beta_k \in \mathbb{D}$ with $\beta_k \neq \alpha_k$. We define as monic COPUC, $\{\Phi_n(z;k)\}_{n\geqslant 0}$, those polynomials generated using (2.7) through the Szego˝ recurrences. Analogously, we denote by $\{\Omega_n(z;k)\}_{n\geqslant 0}$ the corresponding second kind polynomials.

In this point we must emphasize that as a consequence of the CMV representation [20, Chapter 4] and Weyl's theo-rem [20], under a finite composition of the above modifications the essential support of the orthogonality measure remains invariant.

Obviously, $\Phi_n(z;k)$ (resp. $\Omega_n(z;k)$) i s e x a c t l y $\Phi_n(z)$ (resp. $\Omega_n(z)$), except for $n \geqslant k+1$. When $n \geqslant k+1$, we follow an analogue procedure to the one of [20, Chapter 3] to obtain the relation between $\Phi_n(z;k)$ (resp. $\Omega_n(z;k)$) and $\Phi_n(z)$ (resp. $\Omega_n(z)$) using a simple matrix recursion based on the transfer matrix for the COPUC.

Theorem 2.1. *The following relations hold:*

$$2z^{k+2}\kappa_{k+1}^{-2}\begin{bmatrix} -\Omega_{n+1}(z;k) \\ \Phi_{n+1}(z;k)\end{bmatrix} = \mathbf{B}_k(z)\begin{bmatrix} -\Omega_{n+1}(z) \\ \Phi_{n+1}(z)\end{bmatrix}, \quad n \geqslant k+1,$$

where the corresponding transfer matrix $\mathbf{B}_k(z)$ is

$$\mathbf{B}_k(z) = \begin{bmatrix} r(z;k)\Omega_k^*(z) + zr^*(z;k)\Omega_k(z) & s(z;k)\Omega_k^*(z) - zs^*(z;k)\Omega_k(z) \\ r(z;k)\Phi_k^*(z) - zr^*(z;k)\Phi_k(z) & s(z;k)\Phi_k^*(z) + zs^*(z;k)\Phi_k(z)\end{bmatrix},$$

with

$$r(z;k) = (1-\alpha_k\overline{\beta}_k)z\Phi_k(z) - \overline{(\alpha_k - \beta_k)}\Phi_k^*(z),$$
$$s(z;k) = (1-\alpha_k\overline{\beta}_k)z\Omega_k(z) + \overline{(\alpha_k - \beta_k)}\Omega_k^*(z).$$

Proof. Let us introduce the non-singular polynomial matrix

$$\boldsymbol{\Upsilon}_{n+1}(z) = \begin{bmatrix} \Omega_{n+1}^*(z) & \Phi_{n+1}^*(z) \\ -\Omega_{n+1}(z) & \Phi_{n+1}(z)\end{bmatrix}. \tag{2.8}$$

In that way, we can easily check that

$$\boldsymbol{\Upsilon}_{n+1}(z)=\sqrt{2}\boldsymbol{A}_n(z)\boldsymbol{A}_{n-1}(z)\cdots\boldsymbol{A}_0(z)\boldsymbol{J}, \tag{2.9}$$

where the matrices $\boldsymbol{A}_n(z)$ and $\sqrt{2}\boldsymbol{J}$ are

$$\boldsymbol{A}_n(z)=\begin{bmatrix}1 & -\alpha_n z\\ -\overline{\alpha}_n & z\end{bmatrix},\qquad \sqrt{2}\boldsymbol{J}=\boldsymbol{\Upsilon}_0(z)=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}.$$

Notice that $\boldsymbol{J}$ is an orthogonal matrix.

Now, we denote by $\boldsymbol{\Upsilon}_{n+1}(z;k)$ and $\boldsymbol{\Upsilon}^{(k+1)}_{n-(k+1)}(z)$ the polynomial matrices as (2.8) associated with the COPUC and the corresponding associated polynomials of order $k+1$, respectively. Hence, since

$$\boldsymbol{\Upsilon}^{(k+1)}_{n-(k+1)}(z)=\sqrt{2}\boldsymbol{A}_n(z)\boldsymbol{A}_{n-1}(z)\cdots\boldsymbol{A}_{k+1}(z)\boldsymbol{J},$$

we conclude from (2.9) that

$$\boldsymbol{\Upsilon}_{n+1}(z)=\boldsymbol{\Upsilon}^{(k+1)}_{n-(k+1)}(z)\boldsymbol{J}^T\boldsymbol{A}_k(z)\boldsymbol{A}_{k-1}(z)\cdots\boldsymbol{A}_0(z)\boldsymbol{J}, \tag{2.10}$$

$$\boldsymbol{\Upsilon}_{n+1}(z;k)=\boldsymbol{\Upsilon}^{(k+1)}_{n-(k+1)}(z)\boldsymbol{J}^T\boldsymbol{A}_k(z;k)\boldsymbol{A}_{k-1}(z)\cdots\boldsymbol{A}_0(z)\boldsymbol{J}, \tag{2.11}$$

where $\boldsymbol{A}_k(z;k)$ is the matrix containing the perturbed Verblunsky coefficient, i.e.,

$$\boldsymbol{A}_k(z;k)=\begin{bmatrix}1 & -\beta_k z\\ -\overline{\beta}_k & z\end{bmatrix}.$$

Combining (2.10) with (2.11), we get

$$\boldsymbol{\Upsilon}^T_{n+1}(z;k)=\left(\boldsymbol{A}_k(z;k)\boldsymbol{\Upsilon}_k(z)\right)^T\left(\boldsymbol{A}_k(z)\boldsymbol{\Upsilon}_k(z)\right)^{-T}\boldsymbol{\Upsilon}^T_{n+1}(z),$$

where

$$\left(\boldsymbol{A}_k(z;k)\boldsymbol{\Upsilon}_k(z)\right)^T=\begin{bmatrix}Q_{k+1}(z;\beta_k) & -Q^*_{k+1}(z;\beta_k)\\ P_{k+1}(z;\beta_k) & P^*_{k+1}(z;\beta_k)\end{bmatrix},$$

$$2z^{k+2}\kappa^{-2}_{k+1}\left(\boldsymbol{A}_k(z)\boldsymbol{\Upsilon}_k(z)\right)^{-T}=\begin{bmatrix}P^*_{k+1}(z;\alpha_k) & Q^*_{k+1}(z;\alpha_k)\\ -P_{k+1}(z;\alpha_k) & Q_{k+1}(z;\alpha_k)\end{bmatrix},$$

with

$$P_{k+1}(z;\nu)=\Phi^*_k(z)-\nu z\Phi_k(z),\qquad Q_{k+1}(z;\nu)=\Omega^*_k(z)+\nu z\Omega_k(z).$$

An easy computation shows that

$$\left(\boldsymbol{A}_k(z;k)\boldsymbol{\Upsilon}_k(z)\right)^T\left(\boldsymbol{A}_k(z)\boldsymbol{\Upsilon}_k(z)\right)^{-T}=\boldsymbol{B}_k(z), \tag{2.12}$$

which is the desired conclusion. □

Let us consider the simplest case where $k=0$.

Example 2.1 *($k=0$).* In this case, for $i,j=1,2$, $i\neq j$, we get

$$\left(\boldsymbol{B}_0(z)\right)_{i,i}=\left(\boldsymbol{B}_0(z)\right)_{i,i}=(-1)^i(\alpha_0-\beta_0)z^2+2\left(1-\Re(\overline{\alpha}_0\beta_0)\right)z+(-1)^i\overline{(\alpha_0-\beta_0)},$$

$$\left(\boldsymbol{B}_0(z)\right)_{i,j}=\left(\boldsymbol{B}_0(z)\right)_{j,i}=(-1)^i(\alpha_0-\beta_0)z^2-2\Im(\overline{\alpha}_0\beta_0)iz+(-1)^j\overline{(\alpha_0-\beta_0)},$$

where $(\boldsymbol{A})_{i,j}$ denotes the (i,j) entry of the matrix $\boldsymbol{A}$. According to Theorem 2.1, we have

$$2z^2\kappa_1^{-2}\Phi_{n+1}(z;0)=\left(\boldsymbol{B}_0(z)\right)_{2,2}\Phi_{n+1}(z)-\left(\boldsymbol{B}_0(z)\right)_{2,1}\Omega_{n+1}(z),\quad n\geqslant 1,$$

$$2z^2\kappa_1^{-2}\Omega_{n+1}(z;0)=-\left(\boldsymbol{B}_0(z)\right)_{1,2}\Phi_{n+1}(z)+\left(\boldsymbol{B}_0(z)\right)_{1,1}\Omega_{n+1}(z),\quad n\geqslant 1.$$

As expected, when $\beta_0=\alpha_0$, $(\boldsymbol{B}_0(z))_{1,2}=(\boldsymbol{B}_0(z))_{2,1}=0$ and $(\boldsymbol{B}_0(z))_{1,1}=(\boldsymbol{B}_0(z))_{2,2}=2z\kappa_1^{-2}$, we recover the original sequence.

Many classical examples of COPUC can be obtained from finite modifications of well known Verblunsky coefficients. The modification considered in Theorem 2.1 on the Lebesgue polynomials [26,20] yields an interesting sequence of polynomials considered in [14].

Example 2.2 *(T_0-Appell polynomials).* From Theorem 2.1, it is easy to check that the sequence of orthogonal polynomials corresponding to the Verblunsky coefficients $\{\beta_k\delta_{n,k}\}_{n\geqslant 0}$, is given by

$$2z^{k+2}\Xi_{n+1}(z;k) = \left(\left(\boldsymbol{B}_k(z)\right)_{2,2} - \left(\boldsymbol{B}_k(z)\right)_{2,1}\right)z^{n+1}, \quad n \geqslant k+1,$$

where,

$$\left(\boldsymbol{B}_k(z)\right)_{2,1} = -\beta_k z^{2k+2} + \overline{\beta}_k, \qquad \left(\boldsymbol{B}_k(z)\right)_{2,2} = -\beta_k z^{2k+2} + 2z^{k+1} - \overline{\beta}_k.$$

Thus,

$$\Xi_{n+1}(z;k) = z^{n-(k+1)}\left(z^{k+1} - \overline{\beta}_k\right), \quad n \geqslant k+1.$$

The above polynomials are known as T_0-Appell polynomials [14]. When $k=0$ are the Bernstein–Szegő polynomials [20], see also Example 2.1. Indeed, the T_0-Appell polynomials are COPUC associated with the Lebesgue polynomials [26,20].

Next we give a relation between the COPUC associated with two modifications of different level.

Corollary 2.1. *For $\ell < k$, the following relation holds:*

$$z^{k-\ell}\left(\frac{\kappa_{k+1}}{\kappa_{\ell+1}}\right)^{-2}\begin{bmatrix} -\Omega_{n+1}(z;k) \\ \Phi_{n+1}(z;k)\end{bmatrix} = \boldsymbol{B}_k(z)\boldsymbol{B}_\ell^{-1}(z)\begin{bmatrix} -\Omega_{n+1}(z;\ell) \\ \Phi_{n+1}(z;\ell)\end{bmatrix}, \quad n \geqslant k+1.$$

Proof. The proof follows immediately from the representation (2.12). □

For a finite composition of perturbations we have the following result.

Theorem 2.2. *For $0 \leqslant l < \cdots < m < \infty$, the following relation holds:*

$$2^{m-l}\prod_{j=l+1}^{m+1} z^{j+1}\kappa_j^{-2}\begin{bmatrix} -\Omega_{n+1}(z;l,\ldots,m) \\ \Phi_{n+1}(z;l,\ldots,m)\end{bmatrix} = \prod_{j=l}^{m}\boldsymbol{B}_j(z)\begin{bmatrix} -\Omega_{n+1}(z) \\ \Phi_{n+1}(z)\end{bmatrix}, \quad n \geqslant l+1.$$

Proof. Since $\boldsymbol{B}_k(z)$ depends only on the first $k+1$ original Verblunsky coefficients and the perturbed β_k, we have

$$\begin{aligned}
&2z^{m+2}\kappa_{m+1}^{-2}\boldsymbol{\Upsilon}_{n+1}^T(z;m) = \boldsymbol{B}_m(z)\boldsymbol{\Upsilon}_{n+1}^T(z), \quad n \geqslant m+1,\\
&2z^{m+1}\kappa_m^{-2}\boldsymbol{\Upsilon}_{n+1}^T(z;m,m-1) = \boldsymbol{B}_{m-1}(z)\boldsymbol{\Upsilon}_{n+1}^T(z;m), \quad n \geqslant m,\\
&\vdots\\
&2z^{l+2}\kappa_{l+1}^{-2}\boldsymbol{\Upsilon}_{n+1}^T(z;m,\ldots,l) = \boldsymbol{B}_l(z)\boldsymbol{\Upsilon}_{n+1}^T(z;m,\ldots,l-1), \quad n \geqslant l+1.
\end{aligned}$$

On the other hand, it is obvious that $\boldsymbol{\Upsilon}_{n+1}(z;l,\ldots,m) = \boldsymbol{\Upsilon}_{n+1}(z;m,\ldots,l)$. Thus,

$$2^{m-l}\prod_{j=l+1}^{m+1} z^{j+1}\kappa_j^{-2}\boldsymbol{\Upsilon}_{n+1}^T(z;l,\ldots,m) = \prod_{j=l}^{m}\boldsymbol{B}_j(z)\boldsymbol{\Upsilon}_{n+1}^T(z), \tag{2.13}$$

and the result follows. □

When we consider a sequence of perturbed Verblunsky coefficients $\{\lambda_n\alpha_n\}_{n\geqslant 0}$, the resulting sequence is called an Aleksandrov sequence if $|\lambda_n| = 1$, with λ_n a constant complex number. When $|\lambda_n| \neq 1$, with λ_n a non-constant complex number such that $|\lambda_n| < |\alpha_n|^{-1}$ and n belonging to a finite subset of positive integers numbers we have an Aleksandrov-type sequence. This case has not been previously studied in the literature. Notice that in the previous theorem we extend the classical results to the Aleksandrov-type sequence.

As a last example in this section, let us come back to the Aleksandrov sequence using the ideas of Theorem 2.1.

Example 2.3 *(Aleksandrov polynomials).* Let us define the non-singular polynomial matrix

$$\boldsymbol{\Upsilon}_{n+1}^\lambda(z) = \begin{bmatrix} \overline{\lambda}(\Omega_{n+1}^\lambda)^*(z) & \overline{\lambda}(\Phi_{n+1}^\lambda)^*(z) \\ -\Omega_{n+1}^\lambda(z) & \Phi_{n+1}^\lambda(z)\end{bmatrix}, \quad |\lambda| = 1,$$

associated with the Aleksandrov sequence. Here, by using the previous ideas we can obtain the polynomial matrix $\boldsymbol{\Upsilon}_{n+1}^\lambda(z)$ associated with the Aleksandrov sequence. Obviously,

$$\boldsymbol{\Upsilon}_{n+1}^{\lambda}(z)=\boldsymbol{A}_n(z)\cdots\boldsymbol{A}_0(z)\boldsymbol{\Upsilon}_0^{\lambda}(z),$$

where

$$\boldsymbol{\Upsilon}_0^{\lambda}(z)=\begin{bmatrix}\bar{\lambda} & \bar{\lambda}\\ -1 & 1\end{bmatrix}.$$

As in Theorem 2.1, we can see that

$$\left(\boldsymbol{\Upsilon}_{n+1}^{\lambda}\right)^T(z)=\boldsymbol{\Lambda}\boldsymbol{\Upsilon}_{n+1}^{T}(z),\quad \boldsymbol{\Lambda}=\frac{1}{2}\begin{bmatrix}\bar{\lambda}+1 & \bar{\lambda}-1\\ \bar{\lambda}-1 & \bar{\lambda}+1\end{bmatrix}.$$

If $\lambda=-1$, the previous equality yields $\Phi_{n+1}^{\lambda=-1}(z)=\Omega(z)$.

Now, in relation with the COPUC, we are in condition to study one of the most valued objects in the theory of OPUC, the $\mathcal{C}$-functions.

3. Spectral transformations

In this section, for reasons of convenience, we adopt a notation $\doteq$ introduced in [20], i.e., for

$$z=\frac{aw+b}{cw+d},$$

we will write

$$z\doteq\boldsymbol{A}w,\quad \boldsymbol{A}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

The Riesz–Herglotz transformation of the orthogonality measure, $d\sigma(y)$ is

$$F_\sigma(z)=\int_{\partial\mathbb{D}}\frac{y+z}{y-z}\,d\sigma(y),$$

and it has a particular interest in the theory of OPUC. Since $F_\sigma(0)=1$ and $\Re F_\sigma(z)>0$ on $\mathbb{D}$, it is called a Carathéodory function [8], or, simply, a $\mathcal{C}$-function. The analytic function $f_\sigma(z)$ on $\mathbb{D}$ defined by the one-to-one correspondence with $F_\sigma(z)$,

$$zf_\sigma(z):\doteq\boldsymbol{J}^TF_\sigma(z),$$

is called a Schur function or an $\mathcal{S}$-function. Notice that according to this definition, $\sup_{z\in\mathbb{D}}|f_\sigma(z)|\leqslant 1$. On the other hand, if $f_\sigma(z)$ is an $\mathcal{S}$-function, then

$$F_\sigma(z):\doteq\boldsymbol{J}\left(zf_\sigma(z)\right),\tag{3.14}$$

is a $\mathcal{C}$-function.

By a spectral transformation of a $\mathcal{C}$-function $F_\sigma(z)$ we mean a new $\mathcal{C}$-function associated with a measure $d\psi(z)$, a modification of the original measure $d\sigma(z)$. We refer to pure rational spectral transformation as a transformation of $F_\sigma(z)$ given by

$$F_\psi(z)\doteq\boldsymbol{E}(z)F_\sigma(z),\quad \boldsymbol{E}(z)=\begin{bmatrix}A(z) & B(z)\\ C(z) & D(z)\end{bmatrix},\tag{3.15}$$

where $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are non-zero polynomials that provide a 'true' asymptotic behavior to (3.15). In order to $F_\sigma(z)\doteq\boldsymbol{E}^{-1}(z)F_\psi(z)$ we will assume that $\boldsymbol{E}(z)$ is invertible. In a similar way, we can define the pure rational spectral transformations for $\mathcal{S}$-functions.

Notice that the set of rational spectral transformations is not an algebraic group; indeed it is a semigroup. In any case, this is not relevant for the results of the manuscript. We must once again urge the reader to consult [13], where these kinds of transformations for the Stieltjes function (m-function, in short) arise from the study of the COPRL.

3.1. Quadratic irrationality

Let $F(z;k)$ (resp. $f(z;k)$) be the $\mathcal{C}$-function (resp. $\mathcal{S}$-function), associated with the sequence of COPUC $\{\Phi_n(z;k)\}_{n\geqslant 0}$. We begin expressing $F(z;k)$ (resp. $f(z;k)$) in terms of $F_\sigma(z)$ (resp. $f_\sigma(z)$).

Theorem 3.1. *$F(z;k)$ is a pure rational spectral transformation of $F_\sigma(z)$, given by*

$$F(z;k) \doteq \boldsymbol{B}(z;k)F_\sigma(z).$$

Moreover, $f(z;k)$ is a pure rational spectral transformation of $f_\sigma(z)$, given by

$$zf(z;k) \doteq \boldsymbol{D}(z;k)\big(zf_\sigma(z)\big),$$

where

$$\big(\boldsymbol{D}(z;k)\big)_{i,j} = t_{i,j}(z;k) + zt^*_{i+1,j+1}(z;k),$$

with

$$t_{i,j}(z;k) = \big(r(z;k) + (-1)^j s(z;k)\big)\big(\Omega^*_k(z) + (-1)^i \Phi^*_k(z)\big).$$

Proof. From (2.13) we also deduce that

$$2z^{k+2}\kappa_{k+1}^{-2}\begin{bmatrix}\Omega^*_{n+1}(z;k)\\ \Phi^*_{n+1}(z;k)\end{bmatrix} = \boldsymbol{B}_k(z)\begin{bmatrix}\Omega^*_{n+1}(z)\\ \Phi^*_{n+1}(z)\end{bmatrix}, \quad n \geqslant k+1. \tag{3.16}$$

Since, by (3.16),

$$\frac{\Omega^*_{n+1}(z;k)}{\Phi^*_{n+1}(z;k)} = \frac{(\boldsymbol{B}(z;k))_{1,1}\Omega^*_{n+1}(z;k) + (\boldsymbol{B}(z;k))_{1,2}\Phi^*_{n+1}(z;k)}{(\boldsymbol{B}(z;k))_{1,1}\Omega^*_{n+1}(z;k) + (\boldsymbol{B}(z;k))_{1,2}\Phi^*_{n+1}(z;k)},$$

the first part of the theorem follows as a consequence of the Bernstein–Szegő approximation [20, Theorem 1.7.8, Theo-rem 3.2.4].

On the other hand, (3.14) yields

$$zf(z;k) \doteq \boldsymbol{J}^T\boldsymbol{B}_k(z)\boldsymbol{J}\big(zf_\sigma(z)\big),$$

and the last part of the theorem follows. □

From the above theorem, as in [15], we can obtain the orthogonality measure associated with $F(z;k)$. N o w , l e t u s r e t u r n to Example 2.2.

Example 3.1 *(T_0-Appell polynomials).* Let $F_\theta(z) = 1$ b e t h e C-function associated with the Lebesgue polynomials [26,20] and let $F_\delta(z)$ be the C-function associated with the T_0-Appell polynomials given in Example 2.2. S i n c e

$$\big(\boldsymbol{B}_k(z)\big)_{1,1} = \beta_k z^{2k+2} + 2z^{k+1} + \overline{\beta}_k, \qquad \big(\boldsymbol{B}_k(z)\big)_{1,2} = \beta_k z^{2k+2} - \overline{\beta}_k.$$

According to Theorem 3.1 and Example 2.2, w e h a v e

$$F_\delta(z) = \frac{1+\beta_k z^{k+1}}{1-\beta_k z^{k+1}}.$$

From now on, let us focus our attention on the C-functions. Let $F(z;l,\dots,m)$ be the C-function associated with the finite composition of COPUC $\{\Phi_n(z;l,\dots,m)\}_{n\geqslant 0}$. As a consequence of Theorem 2.2, we obtain the next corollary.

Corollary 3.1. *$F(z;l,\dots,m)$ is a pure rational spectral transformation of $F_\sigma(z)$, given by*

$$F(z;l,\dots,m) \doteq \prod_{j=l}^{m}\boldsymbol{B}_j(z)F_\sigma(z).$$

The above corollary can be read in terms of quadratic irrationality. An analytic function $f(z)$ in $\mathbb{D}$ is said to be quadratic irrationality if and only if there exist polynomials $a(z)$, $b(z)$, and $c(z)$ such that

$$a(z)f^2(z) + b(z)f(z) + c(z) = 0, \quad a(z) \neq 0.$$

Lemma 3.1 *(Simon Lemma). (See [20].) Let $f(z)$ and $g(z)$ be analytic in $\mathbb{D}$ such that there exist polynomials $a(z)$, $b(z)$, $c(z)$, and $d(z)$ with*

$$\boldsymbol{A}(z)=\begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}, \qquad \det \boldsymbol{A}(z) \neq 0,$$

so that

$$g(z) \doteq \boldsymbol{A}(z) f(z).$$

Then $g(z)$ is a quadratic irrationality if and only if $f(z)$ is.

As a straightforward consequence of Corollary 3.1 and Lemma 3.1, we obtain the next result.

Corollary 3.2. *$F(z)$ is a quadratic irrationality if and only if $F(z; l, \ldots, m)$ is.*

Notice that Corollary 3.2 characterizes in terms of the Verblunsky coefficients a wide class of orthogonal polynomials such that their corresponding $\mathcal{C}$-functions satisfy a quadratic irrationality. In general terms, this problem was raised in [12, Problem 3] and [20, pp. 791] and still remains open.

A polynomial $P(z)$ is called self-reciprocal if $P(z) = \lambda P^*(z)$, with $|\lambda| = 1$. We would like to draw attention here to an interesting fact. The elements of the matrix $\boldsymbol{B}_k(z)$, in Theorem 3.1, are self-reciprocal polynomials of the same degree satisfying

$$\frac{(\boldsymbol{B}_k(0))_{1,1}}{(\boldsymbol{B}_k(0))^*_{1,1}} = -\frac{(\boldsymbol{B}_k(0))_{1,2}}{(\boldsymbol{B}_k(0))^*_{1,2}} = -\frac{(\boldsymbol{B}_k(0))_{2,1}}{(\boldsymbol{B}_k(0))^*_{2,1}} = \frac{(\boldsymbol{B}_k(0))_{2,2}}{(\boldsymbol{B}_k(0))^*_{2,2}}. \tag{3.17}$$

A natural question is: Are these general properties of the pure rational spectral transformations for $\mathcal{C}$-functions? Let us answer that question in the next section.

3.2. Rational spectral transformations

Before embarking on the next results we need to state a useful characterization due to Peherstorfer and Steinbauer [16].

Theorem 3.2 *(Peherstorfer–Steinbauer Theorem). (See [16].) Let $\pi(z)$ and $\chi(z)$ be polynomials of degree at most n with*

$$\pi(z) F_\sigma(z) + \chi(z) = \mathcal{O}(z^n),$$

and

$$\pi^*(z) F_\sigma(z) - \chi^*(z) = \mathcal{O}(z^{n+1}).$$

Then, $\pi(z) = \Phi_n(z)$ and $\chi(z) = \Omega_n(z)$ with respect to $d\sigma(z)$, up to multiplicative constants.

The preceding theorem has some interesting consequences for the polynomial coefficients associated with (3.15). Indeed,

Theorem 3.3. *Let $F_\sigma(z)$ and $F_\psi(z)$ be $\mathcal{C}$-functions, where $F_\psi(z)$ is the pure rational spectral transformation of $F_\sigma(z)$ given in (3.15). Then, the polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$ are self-reciprocal polynomials of the same degree satisfying*

$$\frac{A(0)}{A^*(0)} = -\frac{B(0)}{B^*(0)} = -\frac{C(0)}{C^*(0)} = \frac{D(0)}{D^*(0)}.$$

Proof. By Theorem 3.2, if we consider $\alpha_{n+1} = \alpha_{n+2} = \cdots = 0$ (Bernstein–Szegő measure), then

$$F_\sigma(z) = \frac{\Omega_n^*(z)}{\Phi_n^*(z)} = -\frac{\Omega_n(z)}{\Phi_n(z)}.$$

Let $\{\gamma_n\}_{n\geqslant 0}$ be the sequence of Verblunsky coefficients associated with $F_\psi(z)$. Considering $\gamma_{n+c+1} = \gamma_{n+c+2} = \cdots = 0$, (3.15) yields

$$-\frac{\Omega_{n+c}(z;\psi)}{\Phi_{n+c}(z;\psi)} = \frac{B(z)\Phi_n(z) - A(z)\Omega_n(z)}{D(z)\Phi_n(z) - C(z)\Omega_n(z)}, \tag{3.18}$$

and

$$\frac{\Omega^*_{n+c}(z;\psi)}{\Phi^*_{n+c}(z;\psi)} = \frac{B(z)\Phi_n^*(z) + A(z)\Omega_n^*(z)}{D(z)\Phi_n^*(z) + C(z)\Omega_n^*(z)}, \tag{3.19}$$

where $c = \max\{\deg A(z), \deg B(z)\} = \max\{\deg C(z), \deg D(z)\}$, and $\Phi_{n+c}(z;\psi)$ and $\Omega_{n+c}(z;\psi)$ are the orthogonal polynomials associated with $F_\psi(z)$ and the corresponding second kind polynomial, respectively. Obviously, there exists a number $\ell \geqslant 0$, such that

$$A(z)F_\sigma(z) + B(z) = \mathcal{O}(z^\ell).$$

Moreover, from Theorem 3.2

$$-\frac{\Omega_{n+\ell}(z)}{\Phi_{n+\ell}(z)} = F_\sigma(z) + \mathcal{O}(z^{n+\ell+1}),$$

and

$$\frac{\Omega^*_{n+\ell}(z)}{\Phi^*_{n+\ell}(z)} = F_\sigma(z) + \mathcal{O}(z^{n+\ell}).$$

Combining the previous equalities, we can state that

$$B(z)\Phi_{n+\ell}(z) - A(z)\Omega_{n+\ell}(z) = \mathcal{O}(z^{\ell+m}), \tag{3.20}$$

and

$$B(z)\Phi^*_{n+\ell}(z) + A(z)\Omega^*_{n+\ell}(z) = \mathcal{O}(z^\ell), \tag{3.21}$$

where $\Phi_{n+\ell}(z) = z^m\Phi_{n+\ell-m}(z)$ with $\Phi_{n+\ell-m}(0) \neq 0$, $m \in \{0, 1, \ldots, n+\ell\}$.

According to (3.18), (3.21) becomes

$$\begin{aligned}B(z)\Phi_{n+\ell}(z) - A(z)\Omega_{n+\ell}(z) &= z^{n+c+\ell}\left(\frac{a_1}{z^\ell} + \frac{a_2}{z^{\ell+1}} + \cdots + \frac{a_{n+c}}{z^{n+c+\ell}}\right)\\ &= a_1 z^{n+c} + a_2 z^{n+c-\ell} + \cdots + a_{n+c},\end{aligned}$$

with $a_1 \neq 0$. Additionally, by (3.20)

$$B(z)\Phi_{n+\ell}(z) - A(z)\Omega_{n+\ell}(z) = b_1 z^{\ell+m} + b_2 z^{\ell+m+1} + \cdots + b_{n+c} z^{n+c+\ell},$$

with $b_1 \neq 0$. Combining these equalities we get

$$B(z)\Phi_{n+\ell}(z) - A(z)\Omega_{n+\ell}(z) = a_1 z^{n+c} + \cdots + b_1 z^{\ell+m}, \tag{3.22}$$

where $a_1 \neq 0$ and $b_1 \neq 0$. Thus, $\deg A(z) = \deg B(z)$. We can now proceed analogously to the proof of $\deg C(z) = \deg D(z)$.

On the other hand, since $\boldsymbol{E}(z)$ is non-singular,

$$F_\sigma(z) \doteq \boldsymbol{E}^{-1}(z)F_\psi(z),$$

and, as above, we can prove that $\deg A(z) = \deg C(z)$ and $\deg B(z) = \deg D(z)$. Thus, $A(z)$, $B(z)$, $C(z)$, and $D(z)$ have the same degree.

Finally, according to the previous result, and considering (3.18) and (3.19), the last part of the theorem follows. □

The above theorem gives a characterization of the polynomial coefficients $A(z)$, $B(z)$, $C(z)$, and $D(z)$ associated with pure rational spectral transformations of $\mathcal{C}$-function. In other words, (3.17) is not an exclusive property of the pure rational spectral transformations for $\mathcal{C}$-functions associated with the finite composition of COPUC. Furthermore, it is clear that our results generalize some ideas contained in [1].

The last question to be discussed will be the relation between the COPRL and the COPUC.

4. Connection with the real line case

In order to get the connection with the real line case, it is necessary to put a natural restriction on the Verblunsky coefficients. It is required that

$$\alpha_n \in (-1, 1),\ n \geqslant 0.$$

There is a relation between the coefficients of the recurrence relations (1.1) and (1.3),

$$a_n = \frac{1}{2}\sqrt{(1+\alpha_{2n-2})(1-\alpha^2_{2n-3})(1-\alpha_{2n-4})}, \quad n \geqslant 0, \tag{4.23}$$

$$b_n = -\frac{1}{2}\alpha_{2n-3}(1+\alpha_{2n-2}) + \frac{1}{2}\alpha_{2n-1}(1-\alpha_{2n-2}), \quad n \geqslant 0. \tag{4.24}$$

This is the so-called Szego˝ transformation [26].

It is easy to check that (2.7) implies through the Szego˝ transformation the modification of both sequences of recurrence coefficients associated with the OPRL. More specifically,

Theorem 4.1. *Let $\{c_n\}_{n\geqslant 0}$ and $\{d_n\}_{n\geqslant 0}$ be the recurrence coefficients for the corresponding COPRL associated with (2.7) through the Szego˝ transformation. Then, for $k=2n-3$ and $k=2n-2$,*

$$c_n^2=\frac{1-\beta_{2n-3}^2}{1-\alpha_{2n-3}^2}a_n^2,$$

$$d_{n-1}=b_{n-1}+\frac{1}{2}(\beta_{2n-3}-\alpha_{2n-3})(1-\alpha_{2n-2}),$$

$$d_n=b_n+\frac{1}{2}(\alpha_{2n-3}-\beta_{2n-3})(1+\alpha_{2n-2}),$$

and

$$c_n^2=\frac{1+\beta_{2n-2}}{1+\alpha_{2n-2}}a_n^2,\qquad c_{n+1}^2=\frac{1-\beta_{2n-2}}{1-\alpha_{2n-2}}a_{n+1}^2,$$

$$d_n=b_n+\frac{1}{2}(\alpha_{2n-3}-\alpha_{2n-1})(\alpha_{2n-2}-\beta_{2n-2}),$$

respectively.

Proof. Using (4.23) and (4.24), the result follows after some elementary calculations. The details are left to the reader. □

Notice that the co-dilated case (resp. co-recursive case) for OPRL implies through the Szegő transformation the modifications of all odd (resp. even) Verblunsky coefficients.

Additionally, there is also a relation between the corresponding m-function $S_\mu(x)$ associated with the measure $d\mu(x)$ and $F_\sigma(z)$, as follows

$$F_\sigma(z)=\frac{1-z^2}{2z}S_\mu(x),$$

or, equivalently,

$$S_\mu(x)=\frac{F_\sigma(z)}{\sqrt{x^2-1}},$$

with $2x=z+z^{-1}$ and $z=x-\sqrt{x^2-1}$.

The next result follows directly from the above equalities.

Theorem 4.2. *Let $S(x;k)$ be the m-functions for the corresponding COPRL associated with (2.7) through the Szego˝ transformation. Then,*

$$\overline{x^2-1}\,S(x;k)\doteq\boldsymbol{B}\big(x-\overline{x^2-1};k\big)\left(\left(\frac{1}{x-\sqrt{x^2-1}}-x\right)S_\mu(x)\right).$$

Finally, an interesting open problem is to analyze the Aleksandrov-type sequences with an infinite set of perturbations as a limit case of our approach. Moreover, as the referee pointed out, the behavior of the zeros is an important aspect when studying the COPRL. A similar investigation would be interesting for the case of the unit circle considering also the corresponding para-orthogonal polynomials on the unit circle.

References

[1] K. Castillo, F. Marcellán, Generators of rational spectral transformations for nontrivial $\mathcal{C}$-functions, Math. Comp. 82 (2013) 1057–1068.
[2] T.S. Chihara, On co-recursive orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 899–905.
[3] T.S. Chihara, An Introduction to Orthogonal Polynomials, Math. Appl., vol. 13, Gordon and Breach, New York–London–Paris, 1978.
[4] J. Dini, P. Maroni, A. Ronveaux, Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux, Port. Math. 46 (1989) 269–282.[5] W. Erb, Optimally space localized polynomials with applications in signal processing, J. Fourier Anal. Appl. 18 (2012) 45–66.
[6] W. Erb, Accelerated Landweber methods based on co-dilated orthogonal polynomials, arXiv:1206.1950, 2013.
[7] W.D. Evans, L.L. Littlejohn, F. Marcellán, C. Markett, A. Ronveaux, On recurrence relations for Sobolev orthogonal polynomials, SIAM J. Math. Anal. 26 (1995) 446–467.
[8] Ya.L. Geronimus, Orthogonal polynomials on the unit circle and their applications, Zap. Nauchn. Issled. Inst. Mat. Mekh. Khar'kov Mat. Obsc. 19 (4) (1948) 35–120; Amer. Math. Soc. Transl. Ser. I, vol. 3, Amer. Math. Soc., Providence, RI, 1962, pp. 1–78.[9] E. Leopold, Perturbed recurrence relations II the general case, Numer. Algorithms 44 (2007) 347–366.

[10] J. Letessier, Some results on co-recursive associated Laguerre and Jacobi polynomials, SIAM J. Math. Anal. 25 (1994) 528–548.
[11] N. Levinson, The Wiener RMS (root-mean square) error criterion in filter design and prediction, J. Math. Phys. Mass. Inst. Tech. 25 (1947) 261–278.
[12] F. Marcellán, Szeg˝o polynomials and Carathéodory functions: Inverse problems, in: B.D. Choi (Ed.), Proceedings of Applied Mathematics Workshop on Analysis, Knots Theory, Probability and Their Applications, Korean Advanced Institute for Science and Technology, Taejon, 1996, pp. 240–252.
[13] F. Marcellán, J.S. Dehesa, A. Ronveaux, On orthogonal polynomials with perturbed recurrence relations, J. Comput. Appl. Math. 30 (1990) 203–212.
[14] F. Marcellán, C. Tasis, Appell orthogonal polynomials on the unit circle, Port. Math. 50 (3) (1993) 335–342.
[15] F. Peherstorfer, A special class of polynomials orthogonal on the unit circle including the associated polynomials, Constr. Approx. 12 (1996) 161–185.
[16] F. Peherstorfer, R. Steinbauer, Characterization of orthogonal polynomials with respect to a functional, J. Comput. Appl. Math. 65 (1995) 339–355.
[17] A. Ronveaux, Fourth-order differential equations for numerator polynomials, J. Phys. A 21 (1988) L743–L753.
[18] A. Ronveaux, S. Belmehdi, J. Dini, P. Maroni, Fourth-order differential equation for the co-modified of semi-classical orthogonal polynomials, J. Math. Anal. Appl. 29 (1990) 225–231.
[19] A. Ronveaux, F. Marcellán, Co-recursive orthogonal polynomials and fourth-order differential equation, J. Comput. Appl. Math. 25 (1989) 105–109.
[20] B. Simon, Orthogonal Polynomials on the Unit Circle, Amer. Math. Soc. Colloq. Publ., vol. 54, Amer. Math. Soc., Providence, RI, 2005, Parts 1, 2.
[21] B. Simon, Szeg˝o's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials, Princeton University Press, Princeton, NJ, 2011.
[22] H.A. Slim, On co-recursive orthogonal polynomials and their application to potential scattering, J. Math. Anal. Appl. 136 (1988) 1–19.[23] G. Szeg˝o, Ein Grenzwertsatz über die Toeplitzschen Determinanten einer reellen positiven Funktion, Math. Ann. 76 (1915) 490–503.[24] G. Szeg˝o, Beiträge zur Theorie der Toeplitzschen formen, Math. Z. 6 (1920) 167–202.
[25] G. Szeg˝o, Beiträge zur Theorie der Toeplitzschen formen II, Math. Z. 9 (1920) 167–190.
[26] G. Szeg˝o, Orthogonal Polynomials, fourth edition, Amer. Math. Soc. Colloq. Publ., vol. 24, Amer. Math. Soc., Providence, RI, 1975.